Optimal Dynamic Pricing of Perishable Products Considering Quantity Discount Policy

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Abstract

The perishable product's dynamic pricing with quantity discount strategy is studied. The seller uses two-period dynamic price during the whole sale period and introduces quantity discount in second sale period in order to improve the seller's profit. The mathematic model expressions are obtained by assuming that consumer's reservation price draws from a uniform distribution and the price discount is the linear function of the consumer's proportion that purchased two units product in the second period. The models provide insights about optimal price and price discount under different levels of the number of the potential consumers, salvage value, reservation price. Finally, Matlab is used to implement the numerical studies and sensitivity tests. We find that the seller's profit increases effectively after using quantity discount strategy through raising the two-period price and lowering the remaining inventory. The effectiveness of the quantity discount depends heavily on the number of potential consumers, the product's salvage value, the difference between highest and lowest reservation price.

Keywords: Nonlinear Programming; Dynamic Pricing; Quantity Discount; Perishable Products

1 Introduction

Dynamic pricing and revenue management have been applied more convenient in many industries such as airlines, hotel and fashion apparel. The common characteristics of these industries include wide range of consumer reservation price, small salvage value after sale season, uncertain demand, fixed capacities, products' perishability and so on. The fixed capacity is proposed by the operational conditions including the long production lead times, relatively short sale season, distant supplier location. Dynamic pricing serves as an effective segmentation mechanism that the seller makes use of the difference in consumer reservation price to maximize expected profits. Sellers set the dynamic price facing uncertain demand in order to reduce the remaining inventory; otherwise the remaining products are cleared out with low salvage value. In this paper, we introduce quantity discount policy in the second period to reduce the remaining inventory in order
to improve seller’s profit. In fact, the quantity discount policy divides the consumer group into more segments.

There are two streams of research that are related to our paper: dynamic pricing in revenue management and quantity discount policy in supply chain coordination. American Airlines defined the objective of yield management (1987) as “to maximize passenger revenue by selling the right seats to the right customers at the right time” [1]. Weatherford and Bodyily [2] (1992) first proposed the perishable product’s dynamic pricing from revenue management’s view, present a taxonomy of the field and an agenda for future work. Until recently, existing literature on dynamic pricing with fixed capacity are extremely widespread. Gallego and van Ryzin [3] (1994) studied the continuous pricing strategy, assuming consumer arrivals to the store follow a stochastic process, they showed that a fixed price policy is asymptotically optimal. When the set of price is finite, the price changes at most once that is asymptotically optimal. Based on Gallego’s and van Ryzin’s paper, Feng and Gallego [4] (1995) investigated the optimal timing of price change. They showed that it is optimal to decrease the price when the time falls below a time threshold, the time benchmark depends on the number of yet unsold items. Bitran and Mondschin [5] (1997) analyzed periodic pricing review policies which prices are allowed to decrease at a discrete intervals of time. They gave the optimal price policy and revenue through empirical analysis. Zhao and Zheng [6] (2000) studied the problem under non- homogeneous demand. They found a closed form solution for the case of a discrete price set. Considering that demand is relevant to price and the remaining inventory, Chatwin [7] (2000) showed the optimal continuous pricing policy in which the form of the demand was assumed to be Possion distribution. Recently, Aviv and Pazgal [8] (2008) studied the optimal pricing when strategic consumer are present. They assumed that consumers’ valuation vary across population and decrease over the sale season. Two pricing strategies were considered: contingent and announced fixed-discount. While the above research try to attract more consumers to make purchase, we use the quantity discount policy attract one consumer to purchase more products.

The vendor uses quantity discount strategy to stimulate the retailer to increase the order quantity in supply chain. Existing research on quantity discount policy include Monahan [9], Chiang, James, Huang et al. [10], Weng [11], Li and Liu [12]. Monahan (1984) put forward quantity discount model form seller’s view, give the optimal discount price. Chiang, James, Huang et al. (1994) analyzed the quantity discount using game theory, numerical experiments verify that quantity discount policy reduce buyer’s cost and increase seller’s profit. Weng (1995) assumed that the demand is the function of price, and obtained quantity discount could stimulate demand. Li and Liu (2006) proved that supply chain performance improved through quantity discount when the demand followed a normal distribution. Above literature present that the quantity discount are used in supply chain to improve buyer’s and seller’s profit. However, there are many examples that quantity discount policy is used between retailers and terminal consumers, the consumer enjoys the discount price when purchase two units or more in shopping mall.

In this paper, we consider a two-period dynamic pricing problem, and introduce all-quantity discount strategy in the second sale period. When the second period’s price discount is the linear function of the increased sales, the model highlights several interesting results about optimal price and price discount under different levels of consumers’ quantity, salvage value, reservation price. This paper is organized as follows. In Section 2, we present the notations and model assumptions. The basic and extension models are present in Section 3. In Section 4, we present the numerical experiments and key findings. Section 5 presents a summary of key results.
2 Problem Assumptions and Model Descriptions

2.1 Notations

The notations commonly used in this paper are listed as follows:

- $c$: the manufacturing cost per item
- $K$: initial inventory at the beginning of the first period
- $T$: sale horizon of length
- $H$: the marking down time
- $p_i$: selling price at market per unit in period $i = 1, 2$ (i.e., retailing price), $p_1 \geq p_2$
- $\delta$: the salvage value per item at the end of the sale horizon, $\delta < c$
- $N$: the number of potential consumers
- $V$: the consumers’ reservation price per item, $V_L$, $V_H$ are the lowest and highest reservation price, respectively
- $F(\cdot)$: distribution function of customers’ reservation price
- $x(i)$: the remaining inventory at the end of $i$ period, $i = 1, 2$

2.2 Model Assumptions

The monopolist who sells the perishable products, has only one opportunity to produce the products because of the long leading time. The sale horizon split into two parts $[0, H]$, $[H, T]$. Assuming that $T$ and $H$ is fixed for the seller. The manufacture has two decisions: a premium price $p_1$ and a discount price $p_2$. Customers arrive to the store at the beginning of the sale season, the customer purchases the product when the reservation price is not less than the current price, or else the customer quits the market without purchasing products and each consumer purchases one item at most. We assume that $F(\cdot)$ is the uniform distribution and the information is symmetric between the seller and consumers, that is, the seller knows the reservation price and consumers know the marking down time and the two-period price.

The seller clears the remaining inventory with $p = \delta$ at the end of the sale horizon. From above assumptions, we can obtain that the sale price is not less than $V_L$. Or else, if $p$ is less than $V_L$, the sales do not increased because customers’ reserve price is not less than $V_L$, but the profit decreases. If the number of the potential consumers is not less than the initial inventory, the seller can sell all the products with $p = V_H - (V_H - V_L)K/N$, so the two-period price is not less than $V_H - (V_H - V_L)K/N$, then the remaining inventory at the end of each sale period is not less than zero.

3 Model Formulation and Solution

In this section, we first present the general formulation of the sales and the profit in the two periods, then discuss the existence and uniqueness of the optimal solutions of both models.
3.1 Basic Model

In this model, the seller set the optimal price $p_1$ and $p_2$ in order to maximize the profit $\Pi(p_1, p_2)$. Customers purchase the product if the sale price is not larger than the customers' reservation price. When the reservation price draws from the uniform distribution, the sales in the first period $N_1=\frac{V_H-p_2}{V_H-V_L}N$, where $p_2 \leq p_1 \leq V_H$; The sales in the second period $N_2=\frac{p_1-p_2}{V_H-V_L}N$, where $p_2 \geq L$, $L=\max\{V_L, V_H-(V_H-V_L)K/N\}$.

Then the seller's profit function is present in Eq. (1)

$$\max \Pi(p_1, p_2) = N_1p_1 + N_2p_2 + x(2)\delta - cK.$$  \hspace{1cm} (1)

subject to

$$p_2 \leq p_1 \leq V_H$$ \hspace{1cm} (2)

$$L \leq p_2$$ \hspace{1cm} (3)

**Theorem 1** If the remaining inventory at the end of the second period $x(2)=0$, then the seller's optimal price $p_1^* = V_H - \frac{(V_H-V_L)K}{2N}$, $p_2^* = V_H - \frac{(V_H-V_L)K}{N}$.

**Proof** If $x(2) = 0$, then $K - \frac{V_H-p_2}{V_H-V_L}N = 0$, so $p_2 = V_H - \frac{(V_H-V_L)K}{N}$. From $x(2) = 0$ we can obtain $K \leq \max = N$, then $p_2 \geq V_L$, the optimal second period price $p_2^* = V_H - \frac{(V_H-V_L)K}{N}$, then the seller’s profit function is $\pi = \frac{V_H-p_1}{V_H-V_L} \cdot Np_1 + \frac{p_1-p_2^*}{V_H-V_L} \cdot Np_2^* - cK$

let $\frac{d\pi}{dp_1} = \frac{N(V_H-p_1)}{(V_H-V_L)} = 0$

substitute $p_2^*$ into the above derivative equation,

then obtain $p_1^* = V_H - \frac{(V_H-V_L)K}{2N}$, where $p_1 \geq p_2$.

From Theorem 1, the seller can choose to sell out the product when $K \leq \max$ with the proper prices; However, it is not necessary to maximize the seller’s profit.

**Theorem 2** If the remaining inventory at the end of the second period $x(2) > 0$, then the optimal price is given

(1) If $\delta \geq \max\{\frac{3V_L-V_H}{2}, V_H - \frac{3K(V_H-V_L)}{2N}\}$, then $p_1^{**} = \frac{2V_H+\delta}{3}$, $p_2^{**} = \frac{V_H+2\delta}{3}$.

(2) If $V_H - \frac{3K(V_H-V_L)}{2N} \leq \delta \leq \frac{3V_H-V_L}{2}$, then $p_1^{**} = \frac{V_H+V_L}{2}$, $p_2^{**} = V_L$

**Proof** If $x(2) > 0$, then $K - \frac{V_H-p_2}{V_H-V_L} \cdot N>0$,

so $p_2 > V_H - \frac{(V_H-V_L)K}{N}$ the profit function

$$\Pi(p_1, p_2) = \frac{V_H-p_1}{V_H-V_L} \cdot Np_1 + \frac{p_1-p_2}{V_H-V_L} \cdot Np_2 + x(2)\delta - cK,$$

where $x(2) = K - \frac{V_H-p_2}{V_H-V_L} \cdot N$.

Let the partial derivative of $p_1, p_2$ equal to zero,

$$\frac{\partial \pi}{\partial p_1} = \frac{N(V_H-2p_1+p_2)}{(V_H-V_L)} = 0$$

$$\frac{\partial \pi}{\partial p_2} = \frac{N(p_1-2p_2+\delta)}{(V_H-V_L)} = 0$$
\[ p_1' = \frac{2V_H + \delta}{3}, \quad p_2' = \frac{V_H + 2\delta}{3} \]

(1) If \( \delta \geq \frac{3V_H - V_L}{2} \), then \( p_2 > V_L \), so the optimal price \( p_1'' = \frac{2V_H + \delta}{3} \), \( p_2'' = \frac{V_H + 2\delta}{3} \).

(2) If \( \delta < \frac{3V_H - V_L}{2} \), then the optimal price \( p_1'' = \frac{V_H + V_L}{2} \), \( p_2'' = V_L \).

Theorem 2 presents the relationship between price and salvage value when the remaining inventory is larger than zero. If the salvage value is big enough (\( \delta \geq \frac{3V_H - V_L}{2} \)), the seller maximize his profit by setting higher price, even though there are some remaining inventory; Or else, the seller will meet all the demand with \( p_2'' = V_L \).

**Corollary 1** If the initial inventory is \( K \), when the lowest and highest reservation price are determined, the seller’s optimal price strategy \((\hat{p}_1, \hat{p}_2) = \arg \max \{\Pi(p_1', p_2'), \Pi(p_1'', p_2'')\}\).

It is a game between price and remaining inventory for the seller to maximize his profit. The relative high price corresponding to numerous remaining inventory; The lower price corresponding to less remaining inventory. The seller how to choose the price to balance the remaining inventory, Theorem 1 and Theorem 2 present the answer.

### 3.2 Extension Model

In this section, we introduce the all-quantity discount strategy in second phase to increase the sales in order to improve the seller’s profit, considering that the remaining inventory will be cleared out with salvage price \( \delta \) after the sale season \( T \).

There are two cases for the seller to utilize the all-quantity discount: Case 1: The seller has the remaining inventory at the end of the second sale period in the basic model; Case 2: The seller intends to improve profit through raising the first period price, even though the seller doesn’t have the remaining inventory at the end of the second sale period in the basic model.

We assume that the cost that is incurred by using the quantity discount strategy is zero. Customers purchase one or two units item. If a consumer purchases two units item at a time, he will enjoy the discount price \( \theta p_2 \) with the two products, where \( \theta \) is the price discount factor and \( \theta \in [c/p_2, 1] \). Let \( \alpha \) is the proportion of consumers who purchased two units product in second sale period, then \( \theta \) is the decreasing function of \( \alpha \). Assuming that \( \alpha \) is the linear function of \( \theta \), then there is a reverse function \( \theta = f(\alpha) \), where \( \alpha \in [0, f^{-1}(c/p_2)] \). The seller set the two-period price, price discount factor according to the number of customers and reservation price’s distribution to maximize seller’s profit. The remaining inventory with quantity discount is denoted by \( x'(t) \) at the end of \( t \) period.

Denoting \( N_1 \) is the sales in \([0, H] \) with \( p_1 \), then \( N_1 = \frac{V_H - p_1}{V_H - V_L} N, \quad p_2 \leq p_1 \leq V_H \).

Denoting \( N_{21}, N_{22} \) are the sales in \([H, T] \) with \( p_2, \theta p_2 \), respectively. The total sales in second sale period \( N_2 = N_{21} + N_{22} \). From the above assumptions, we obtain \( N_{21} = \frac{p_2 - p_1}{V_H - V_L} (1 - \alpha) N, \quad N_{22} = \frac{2(p_1 - p_2)}{V_H - V_L} \alpha N, \quad \theta p_2, \quad p_2 \geq L, \quad L = \max \{V_L, V_H - (V_H - V_L) K/N\} \).

Considering \( x'(2) \geq 0, \quad x'(2) = K - \frac{N(V_H - p_2) + N_2 p_2}{(V_H - V_L)}, \) we can obtain \( \alpha \) satisfying \( \alpha \leq s_1 \), then \( \alpha \in [0, s_1] \cap [0, s_2] \), where \( s_1 = \frac{K(V_H - V_L) - N(V_H - p_2)}{N(p_1 - p_2)}, \quad s_2 = f^{-1}(c/p_2) \).

The seller’s profit function with quantity discount is

\[
\max \Pi_1(p_1, p_2, \theta) = N_1 p_1 + N_{21} p_2 + N_{22} \theta p_2 + x'(2) \delta - cK
\]
subject to
\[ p_2 \leq p_1 \leq V_H \]  
\[ L \leq p_2 \]  
\[ L \leq \theta p_2 \]  
\[ \alpha \in [0, s_1] \cap [0, s_2] \]

Theorem 3 The optimal price strategy \((\hat{p}_1, \hat{p}_2)\) and the price discount factor \(\theta^*\) exist when the seller uses the all-quantity discount in the second sale period.

Proof The objective function is continuous function and the feasible region is closed set, then the maximum must exist.

The above problem is nonlinear programming, so the solution can be obtained by solving the K-T condition; However, the explicit solution cannot be expressed because of the complexity of the problem. Therefore, the numerical examples below are calculated by Matlab.

4 Numerical Experiments

The proportion of consumers who purchased two units product in the second sale period is the decreasing function of the price discount factor. Then we assume that the reverse function is \(\theta = 1 - \alpha\), where \(\alpha \in [0, s_1] \cap [0, s_2]\), \(s_1 = \frac{K(V_H - V_L) - N(V_H - p_2)}{N(p_1 - p_2)}\), \(s_2 = 1 - c/p_2\). Assuming that \(K = 50, c = 2\) in the below examples. All the solutions of the basic model and the extension model present in this section are calculated by Matlab 6.5.

4.1 Numerical Examples

Let \(V_H = 6, V_L = 2, \delta = 0.5\), \((\Pi_1 - \Pi)/\Pi\) represents the profit increment rate, it reflects the effectiveness of the quantity discount strategy; Table 1 presents the change of prices, price discount factor and profits when the number of potential consumer decreases.

From Table 1, the seller’s two-period price and profit with quantity discount strategy increase when the quantity discount policy is used in second period. There are three main ways to use quantity discount strategy to raise the seller’s profit. Firstly, the seller can raise two-period price when the number of potential consumer is much larger than initial inventory \((55 \leq N \leq 60)\); Secondly, the seller not only raises the two-period price but also lowers the remaining inventory by using quantity discount policy when the number of potential consumer is much smaller than initial inventory \((40 \leq N \leq 51)\); Finally, the seller could lower remaining inventory effectively by using quantity discount policy when the initial inventory is close to the number of potential consumer \((52 \leq N \leq 54)\).

The optimal two-period price lowers as the number of potential consumers decreases when the optimal remaining inventory equals to zero. The optimal two-period price has no change as the number of potential consumer decreases when the optimal remaining inventory is not zero. Take the seller doesn’t use quantity discount for an example, there are two reasons for the unchanged
Table 1: Optimal price, discount factor and profit without and with quantity discount as a function of $N$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$(\hat{p}_1, \hat{p}_2)$</th>
<th>$x(2)$</th>
<th>$\Pi$</th>
<th>$(\hat{p}_1, \hat{p}_2)$</th>
<th>$\theta^*$</th>
<th>$\theta^* p_2$</th>
<th>$x'(2)$</th>
<th>$\Pi_1$</th>
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<tr>
<td>60</td>
<td>(4.33, 2.67)</td>
<td>0</td>
<td>75.00</td>
<td>(4.51, 2.90)</td>
<td>0.86</td>
<td>2.48</td>
<td>0</td>
<td>78.04</td>
</tr>
<tr>
<td>55</td>
<td>(4.18, 2.36)</td>
<td>0</td>
<td>63.64</td>
<td>(4.40, 2.66)</td>
<td>0.83</td>
<td>2.21</td>
<td>0</td>
<td>67.59</td>
</tr>
<tr>
<td>54</td>
<td>(4.17, 2.33)</td>
<td>0.50</td>
<td>61.13</td>
<td>(4.38, 2.61)</td>
<td>0.83</td>
<td>2.15</td>
<td>0</td>
<td>65.29</td>
</tr>
<tr>
<td>53</td>
<td>(4.17, 2.33)</td>
<td>1.42</td>
<td>58.60</td>
<td>(4.36, 2.55)</td>
<td>0.82</td>
<td>2.09</td>
<td>0</td>
<td>62.92</td>
</tr>
<tr>
<td>52</td>
<td>(4.17, 2.33)</td>
<td>2.33</td>
<td>56.08</td>
<td>(4.34, 2.50)</td>
<td>0.81</td>
<td>2.03</td>
<td>0</td>
<td>60.47</td>
</tr>
<tr>
<td>51</td>
<td>(4.17, 2.33)</td>
<td>3.25</td>
<td>53.56</td>
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<td>0.81</td>
<td>2</td>
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<tr>
<td>50</td>
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<td>51.04</td>
<td>(4.33, 2.46)</td>
<td>0.81</td>
<td>2</td>
<td>1.3947</td>
<td>55.31</td>
</tr>
<tr>
<td>49</td>
<td>(4.17, 2.33)</td>
<td>5.08</td>
<td>48.52</td>
<td>(4.33, 2.46)</td>
<td>0.81</td>
<td>2</td>
<td>2.37</td>
<td>52.71</td>
</tr>
<tr>
<td>48</td>
<td>(4.17, 2.33)</td>
<td>6.00</td>
<td>46.00</td>
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<td>0.81</td>
<td>2</td>
<td>3.34</td>
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<td>38.44</td>
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<tr>
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<td>13.33</td>
<td>25.83</td>
<td>(4.33, 2.46)</td>
<td>0.81</td>
<td>2</td>
<td>11.12</td>
<td>29.25</td>
</tr>
</tbody>
</table>

price: Firstly, the seller’s profit doesn’t increase if the seller set lower price to clear the remaining inventory when the number of potential consumer is relative large ($50 \leq N \leq 54$); Secondly, there is no more consumer to purchase the product if the seller lower price when the number of potential consumer is much smaller ($40 \leq N \leq 49$). Therefore, the optimal two-period price is no change when the number of potential consumers decreases to a certain extent ($N = 54$).

The optimal price discount factor is determined by $s_1$ when the optimal remaining inventory equals to zero ($52 \leq N \leq 60$); Otherwise, the optimal price discount factor is determined by $s_2$. So the optimal price discount factor has no change when ($40 \leq N \leq 51$).

Let $V_H = 6$, $V_L = 2$, $\delta = 0.25c$, $\delta = 0.50c$, $\delta = 0.75c$, we can obtain Fig. 1 by solving the extension model.

From Fig. 1, we can see that the effectiveness of the quantity discount increases as the number

![Fig. 1: The rate of increased profit as a function of $V_H$](image-url)
of the potential consumers and product’s salvage value decrease. The maximum profit increased rate is 0.06 when \( \delta = 0.25c \) with quantity discount; Such as the fashion apparel in Christmas, the perishable fresh food are very suitable to take quantity discount policy.

4.2 Sensitivity Analysis

The lowest and highest reservation price that the seller estimates may not accurate, we assume that the lowest reservation price fluctuates between 1.5 and 2.5 when \( V_H = 6, \delta = 0.5 \), the highest reservation price between 5.5 and 6.5 when \( V_L = 2, \delta = 0.5 \). Then we obtain Fig. 2, Fig. 3 by solving the basic and extension models using Matlab.

From Fig. 2, we can find that the optimal two-period price, discount price and the seller’s profit increase when the highest reservation price \( V_H \) increases. But the price discount factor decreases as \( V_H \) increases, the reason is that the difference increases between \( V_H \) and \( V_L \), hence the sales increases obviously if the discount factor is small in the second period.

From Fig. 3, we can find that the optimal two-period price, discount price, price discount factor and seller’ profit increase when the lowest reservation price \( V_L \) increases. The price discount factor increases because the difference between \( V_H \) and \( V_L \) decreases as \( V_L \) increases, then the sales increases obviously in second period when the price changes from \( p_1 \) to \( p_2 \). Therefore, it is not necessary for the seller to improve sales by lowering price discount factor at the moment.

From Fig. 2, Fig. 3, we can obtain that the bigger the difference between \( V_H \) and \( V_L \), the more effective when the seller uses quantity discount strategy.

Fig. 2: The optimal pricing policy, price discount factor and profit as a function of \( V_H \)
5 Summary

This paper studies the optimal two-period dynamic pricing strategy when the all-quantity discount strategy is introduced in the second sale period motivated by the actuality of the perishable products. We consider two models: the basic model and the extension model. We analyze the impact of the number of potential consumers, the consumers’ reservation price and the product’s salvage value on the price, price discount factor and the profit through the numerical experiments. The key findings are: The seller’s profit increases effectively after using quantity discount strategy. There are main two ways to improve profit for the seller: Firstly, improving the two-period price; Secondly, lowering remaining inventory. The seller’s profit, the two-period price and price discount decrease when potential consumers’ quantity decreases. However, the two-period price and price discount no longer change when the number of potential consumers decreases to a certain extent. We show that the profit increment rate increases when the product’s salvage value lowers or the number of potential consumers decreases. The seller’s profit and two-period price increase when the lowest or highest reservation price increases. However, the price discount factor increases with the increasing of the lowest reservation price or the decreasing of the highest reservation price. The profit increment rate increases obviously when the difference between highest and lowest reservation price varies significantly.

References


