New Explicit Jacobi Elliptic Function Solutions for the Zakharov Equations

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Abstract

In this letter, some new complex doubly periodic solutions to the Zakharov equations are obtained by using the generalized Jacobi elliptic function expansion method with the aid of mathematica software, some of which are degenerated to the solitary wave solutions and the triangle function solutions in the limit cases when the modulus of the Jacobian elliptic functions \( m \to 1 \) or 0, which shows that the general method is more powerful and will be used in further works to establish more entirely new solutions for other kinds of nonlinear partial differential equations arising in mathematical physics.

Keywords: Zakharov Equations; Generalized Jacobi Elliptic Function Expansion Method; Solitary Wave Solutions; Exact Solutions

1 Introduction

Looking for exact solutions to Nonlinear Evolution Equations (NEEs) has long been a major concern for both mathematicians and physicists. Many powerful methods have been developed in these years, such as inverse scattering transformation, Hirota bilinear method, Backlund transformation, Darboux transformation, homogeneous balance method, Lie group analysis, similarity reduced method, F-expansion method and so on. In the interaction of laser-plasma, the System of Zakharov Equation (SZE) plays an important role \cite{1, 2}. This system attracted many scientists’ wide interest and attention. In one dimension, the formation, evolution and interaction of the Langmuir solution differ from solutions of the KdV equation. In multi-dimensions, the Langmuir solution will collapse. Since 1980s, the effects including magnetic field have been considered, and the system of Zakharov equation includes more general form and rich contents. For example, SZE with Landau damping effect was given in \cite{3, 4}. Under some conditions its inverse scattering transformation has been found. In \cite{4}, the longitudinal and transverse oscillating and magnetic

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field effect was examined, the solution properties and collapse in multi-dimensions have been revealed.

More recently, some authors considered the exact and explicit solutions of the system of Zakharov equations by different methods in [5-16]. In this paper, we consider the system of Zakharov equations by constructing four new types of Jacobian elliptic functions, and abundant new families of exact solutions are obtained.

The aim of this paper is to find the new and more general exact Jacobian elliptic functions solutions of Eq. (1) by using the new generalize Jacobi elliptic function expansion method which was first mentioned by Hong [17]. The character feature of our method is that, without much extra effort, we can get series of exact solutions using an uniform way. Another advantage of our method is that it also applies to general nonlinear differential equations.

2 Summary of the Generalized Jacobi Elliptic Functions Expansion Method

For a given partial differential equation in two variables $x$ and $t$

$$P(u, ut, u_x, u_{xt}, u_{tt}, u_{xx}, \cdots) = 0,$$

We seek the following formal solutions of the given system:

$$u(\xi) = A_0 + \sum_{i,j=1, i \leq j \leq n} [A_i F^{i}(\xi) + B_j F^{j-i}(\xi) E^{i}(\xi) + C_i F^{i}(\xi) G^{i}(\xi) + D_i F^{j-i}(\xi) H^{i}(\xi)],$$

where $A_0, A_i, B_i, C_i, D_i (i = 1, 2, \cdots, n)$ are constants to be determined later. $\xi = \xi(x, t)$ are arbitrary functions with the variables $x$ and $t$. The parameter $n$ can be determined by balancing the highest order derivative terms with the nonlinear terms in Eq. (1). And $E(\xi), F(\xi), G(\xi), H(\xi)$ are an arbitrary array of the four functions $e = e(\xi), f = f(\xi), g = g(\xi)$ and $h = h(\xi)$, the selection obey the principle which makes the calculation more simple. Here we ansatz

$$\begin{cases}
  e = \frac{1}{p + qsn\xi + rcn\xi + ldn\xi}, f = \frac{sn\xi}{p + qsn\xi + rcn\xi + ldn\xi}, \\
  g = \frac{cn\xi}{p + qsn\xi + rcn\xi + ldn\xi}, h = \frac{dn\xi}{p + qsn\xi + rcn\xi + ldn\xi},
\end{cases}$$

where $p, q, r, l$ are arbitrary constants which ensure denominator unequal to zero, so do the following situations. The four function $e, f, g, h$ satisfy the following relations

$$\begin{cases}
  e' = -qgh + rfh + lm^2 fg, f' = pgg + rhg + leg, \\
  g' = -pfh - qeh + l(m^2 - 1) ef, h' = -m^2 pfg - r(m^2 - 1) ef - qeg,
\end{cases}$$

where $\frac{d}{d\xi}$ denotes $\frac{d}{d\xi}$, $m$ is the modulus of the Jacobi elliptic function $(0 \leq m \leq 1)$, and $e, f, g, h$ satisfy one of the following relations at the same time.

**Family 1:** When $p = 0$, we can select $F(\xi) = f(\xi)$ or $F(\xi) = g(\xi)$, using the restriction:

$$\begin{cases}
  lh = 1 - qf - rg, e^2 = f^2 + g^2, \\
  (l^2 - r^2) g^2 = 1 - 2(qf + rg - qrg) + (l^2 m^2 - l^2 + q^2) f^2.
\end{cases}$$

(5a)
Family 2: When \( q = 0 \), we can select \( F(\xi) = g(\xi) \) or \( F(\xi) = h(\xi) \), using the restriction:

\[
\begin{align*}
pe & = 1 - rg - lh, (m^2 - 1)f^2 = g^2 - h^2, \\
(l^2(m^2 - 1) + p^2)h^2 & = (1 - m^2)(1 - 2lh + rg - rglh) + r^2g^2 + m^2p^2g^2.
\end{align*}
\]

(5b)

Family 3: When \( r = 0 \), we can select \( F(\xi) = h(\xi) \) or \( F(\xi) = e(\xi) \), using the restriction:

\[
\begin{align*}
qf & = 1 - pe - lh, m^2g^2 = h^2 + (m^2 - 1)e^2, \\
(q^2 - m^2p^2)e^2 & = m^2 - 2m^2lh + pe - pleh + (l^2m^2 + q^2)h^2.
\end{align*}
\]

(5c)

Family 4: When \( l = 0 \), we can select \( F(\xi) = e(\xi) \) or \( F(\xi) = f(\xi) \), using the restriction:

\[
\begin{align*}
rg & = 1 - pe - qf, h^2 = e^2 - m^2f^2, \\
(q^2 + r^2)f^2 & = -1 + 2(pe + qf - pqef) + (r^2 - p^2)e^2.
\end{align*}
\]

(5d)

Substituting (4) along with (5a-5d) into Eq. (1) separately yields four families of polynomial equations for \( E(\xi), F(\xi), G(\xi), H(\xi) \). Setting the coefficients of \( F^i(\xi)E(\xi)^jG(\xi)^kH(\xi)^l \) \((i = 0, 1, 2, \cdots) \) \((j_{i \cdots 3} = 0, 1; j_{1, j_{2, j_{3}}} = 0) \) to zero yields a set of Over-determined Differential Equations (ODEs) in \( A_0, A_1, B_1, C_1, D_1 \), \((i = 1, 2, \cdots, n) \) and \( \xi(x, t) \), solving the ODEs by Mathematica and Wu elimination, we can obtain exact solutions of Eq. (1) according to (2) (3). Obviously, if we choose the special value of \( p, q, r, l \) and \( m \) in (3), then we can get the results in [18-24], which has been discussed in [17, 25]. This method can be used to extend many other traditional methods [26, 27].

3 Exact Solutions to the Zakharov Equations

We consider the following system of Zakharov equations [1-16]

\[
\begin{align*}
u_{tt} - c_s^2u_{xx} - \beta(|v|^2)_{xx} & = 0, \\
i v_t + \alpha v_{xx} - \delta uv & = 0.
\end{align*}
\]

(6)

where \( u = u(x, t) \) is the perturbed number density of the ion (in the low-frequency response), \( v = v(x, t) \) is the slow variation amplitude of the electric field intensity, \( c_s \) is the thermal transportation velocity of the electron-ion, \( \alpha \neq 0, \beta \neq 0, \delta \neq 0, c_s \) are constants. Eqs. (6a) and (6b) are one of the fundamental models governing dynamics of nonlinear waves, and describe the interactions between high- and low-frequency waves. The physically most important example involves the interaction between Langmuir and ion-acoustic waves in plasma.

Since \( v(x, t) \) is a complex function, thus we introduce a gauge transformation

\[
\begin{align*}
u & = u(x, t) = u(\xi), \\
v & = v(x, t) = v(\xi) = \phi(\xi) \exp[i(sx - \omega t)], \\
\xi & = k(x - ct) + \xi_0.
\end{align*}
\]

(7)
where \( \phi(\xi) \) is a real-valued function, \( s, \omega, k, c \) are four real constants to be determined and \( \xi_0 \) is an arbitrary constant. Substituting (8) into (6), we have

\[
\begin{cases}
  k^2(c^2 - c_s^2)u'' - \beta k^2(\phi')'' = 0, \\
  \alpha k^2 \phi'' + (\omega - \alpha s^2) \phi - \delta u \phi + i(2\alpha k - kc) \phi' = 0,
\end{cases}
\]

integrating equation (8a) twice with respect to \( u \) and put the integration constants to zero, we obtain

\[ u = \frac{\beta}{c^2 - c_s^2} \phi^2. \quad (9) \]

Let \( c = 2\alpha s \) and substituting (9) into (8b), we obtain

\[ \alpha k^2 \phi'' + (\omega - \alpha s^2) \phi - \frac{\delta \beta}{c^2 - c_s^2} \phi^3 = 0. \quad (10) \]

By the homogenous balance principle we have \( n = 1 \), thus we assume that the Lie\n
\[ g \text{-}\text{ordan equation} \]

\[ (10) \] have the following solutions

\[ \phi = c_0 + c_1 e + c_2 f + c_3 g + c_4 h, \quad (11) \]

where \( \phi = \phi(\xi), e = e(\xi), f = f(\xi), g = g(\xi), h = h(\xi) \) and \( e, f, g, h \) satisfy (4) and (5a-5d). Substituting (4) and (5a-5d) along with (7c) and (11) into (10) and setting the coefficients of \( F^n(\xi), F^n(\xi) E(\xi), F^n(\xi) G(\xi), F^n(\xi) H(\xi), F^n(\xi) E(\xi)G(\xi), E(\xi)H(\xi), G(\xi)H(\xi), H(\xi), F^n(\xi) G(\xi)H(\xi), \]

\[ i = 0, 1, 2, 3, 4, p, q, r, l. \] We could determine the following solutions of Zakharov equations (6)

**Case 1**

\[ p = 0, q = \sqrt{1 - m^2}, l = 1, r = \mp i, i = \sqrt{-1}, \omega = (k^2(1 - \frac{m^2}{2}) + s^2)\alpha \]

\[ c_0 = \varepsilon k m \sqrt{\frac{\alpha(c^2 - c_s^2)}{2\varepsilon \beta}}, \varepsilon^2 = 1, c_1 = c_2 = c_4 = 0, c_3 = \pm 2c_0 i \]

**Case 2**

\[ \omega = (k^2(1 - \frac{m^2}{2}) + s^2)\alpha, c_0 = c_2 = c_3 = 0, p = l = 0, \]

\[ c_1 = \pm k \sqrt{\frac{\alpha(q^2 + (1 - m^2)r^2)(c^2 - c_s^2)}{2\varepsilon \beta}}, c_4 = \varepsilon k \sqrt{\frac{\alpha(q^2 + r^2)(c^2 - c_s^2)}{2\varepsilon \beta}} \]

We acquire the following Jacobi elliptic function solution

\[ v_1 = \left[ \varepsilon k m \sqrt{\frac{\alpha(c^2 - c_s^2)}{2\varepsilon \beta}} \pm \frac{2k m i}{\sqrt{1 - m^2} \sin \xi_1 + i \cos \xi_1 + \sin \xi_1 + i \cos \xi_1} \right] \exp \left\{ i(s x - (k^2(1 - \frac{m^2}{2}) + s^2) \alpha t) \right\} \]

\[ u_1 = - \left[ \varepsilon k m \sqrt{\frac{\alpha}{2\varepsilon \beta}} \pm \frac{2k m i}{\sqrt{1 - m^2} \sin \xi_1 + i \cos \xi_1 + \sin \xi_1 + i \cos \xi_1} \right]^2 \]

\[ \xi_1 = k(x - ct) + \xi_0 \]

\[ v_2 = k \sqrt{\frac{\alpha(c^2 - c_s^2)}{2\varepsilon \beta}} (\pm \sqrt{q^2 + (1 - m^2)r^2} + \sqrt{q^2 + r^2}) \exp \left\{ i(s x - (k^2(1 - \frac{m^2}{2}) + s^2) \alpha t) \right\} \]

\[ u_2 = \frac{ak^2}{2\varepsilon} (\pm \sqrt{q^2 + (1 - m^2)r^2} + \sqrt{q^2 + r^2}) \exp \left\{ i(s x - (k^2(1 - \frac{m^2}{2}) + s^2) \alpha t) \right\} \]

\[ \xi_2 = k(x - ct) + \xi_0 \]
Case 3

\[ q = 0, p = \sqrt{1 - m^2}, l = 1, r = m, \omega = (s^2 - \frac{1}{2}k^2(1 + m^2))\alpha, \]
\[ c_0 = \epsilon k \sqrt{\frac{\alpha(c^2 - c^2)(m^2 - 1)}{2\beta \delta}}, \epsilon^2 = 1, c_1 = c_2 = c_4 = 0, c_3 = \pm 2c_0 m \]

Case 4

\[ \omega = -\frac{1}{2}(k^2(1 + m^2) - 2s^2)\alpha, c_0 = c_1 = c_2 = c_4 = 0, c_3 = \pm kp \sqrt{\frac{\alpha(1 - m^2)(c^2 - c^2)}{2\beta \delta}}, q = \epsilon p, r = l = 0 \]

Case 5

\[ \omega = \left(k^2 \left(m^2 - \frac{1}{2}\right) + s^2\right)\alpha, c_0 = c_1 = c_3 = c_4 = 0, c_2 = \pm kp \sqrt{\frac{\alpha(c^2 - c^2)}{2\beta \delta}}, r = \epsilon p, q = l = 0 \]

We acquire the following Jacobi elliptic function solutions

\[ v_3 = \left[ \epsilon k \sqrt{\frac{\alpha(c^2 - c^2)(m^2 - 1)}{2\beta \delta}} \pm 2\epsilon km \sqrt{\frac{\alpha(c^2 - c^2)(m^2 - 1)}{2\beta \delta}} \right] \exp[i(sx - (s^2 - \frac{1}{2}k^2(1 + m^2))\alpha t)] \]
\[ u_3 = -\left[ \epsilon k \sqrt{\frac{\alpha(m^2 - 1)}{2\delta}} \pm 2\epsilon km \sqrt{\frac{\alpha(m^2 - 1)}{2\delta}} \right]^2 \]
\[ \xi_3 = k(x - ct) + \xi_0 \]
\[ v_4 = \pm k \sqrt{\frac{\alpha(m^2 - 1)}{2\delta}} \frac{\cn \xi_4}{1 + \epsilon \sn \xi_4} \exp[i(sx + \frac{1}{2}(k^2(m^2 - 2s^2))\alpha t)] \]
\[ u_4 = \frac{\alpha k^2 (1 - m^2)}{2\beta \delta} \frac{\cn \xi_4}{1 + \epsilon \sn \xi_4}^2 \]
\[ \xi_4 = k(x - ct) + \xi_0 \]
\[ v_5 = \pm k \sqrt{\frac{\alpha(c^2 - c^2)}{2\beta \delta}} \frac{\sn \xi_5}{1 + \epsilon \cn \xi_5} \exp[i(sx - (k^2(m^2 - \frac{1}{2}) + s^2))\alpha t)] \]
\[ u_5 = \frac{\alpha k^2 (1 - m^2)}{2\beta \delta} \frac{\sn \xi_5}{1 + \epsilon \cn \xi_5}^2 \]
\[ \xi_5 = k(x - ct) + \xi_0 \]

Case 6

\[ r = 0, p = 0, q = 1, l = 1, \omega = (s^2 + 2k^2(m^2 - \frac{1}{2}))\alpha, \]
\[ c_1 = \pm km \sqrt{\frac{\alpha(c^2 - c^2)}{2\beta \delta}}, c_0 = c_2 = c_4 = 0, c_3 = \epsilon k \sqrt{\frac{\alpha(c^2 - c^2)(1 + m^2)}{2\beta \delta}}, \epsilon^2 = 1 \]

Case 7

\[ \omega = (k^2(m^2 - \frac{1}{2}) + s^2)\alpha, c_0 = c_1 = c_2 = c_3 = c_4 = 0, c_2 = \pm kp \sqrt{\frac{\alpha(c^2 - c^2)}{2\beta \delta(m^2 - 1)}}, q = l = 0, r = \frac{\epsilon p}{\sqrt{m^2 - 1}} \]

Case 8

\[ l = 0, p = q = r = 1, \omega = \frac{1}{2}(s^2 - k^2(1 + m^2))\alpha, \]
\[ c_0 = \mp km \sqrt{\frac{\alpha(c^2 - c^2)(1 - m^2)}{2\beta \delta}}, c_1 = c_3 = c_4 = 0, c_2 = \pm k \sqrt{\frac{3\alpha(c^2 - c^2)(1 - m^2)}{\beta \delta}} \]
We acquire the following Jacobi elliptic function solutions

\[ v_6 = \left[ \pm km \sqrt{\frac{\alpha(c^2-c_0^2)}{2\beta^3}} + \pm k^2 \sqrt{\frac{2 \alpha(c^2-c_0^2)(1-m^2)}{1+sn \xi_0 + cn \xi_0}} \right] \exp[i(sx - (s^2 + k^2(m^2 - \frac{1}{2}))\alpha t)] \]

\[ u_6 = \left[ \pm km \sqrt{\frac{\alpha(c^2-c_0^2)}{2\beta^3}} + \pm k^2 \sqrt{\frac{2 \alpha(c^2-c_0^2)(1-m^2)}{1+sn \xi_0 + cn \xi_0}} \right]^2 \]

\[ \xi_6 = k(x - ct) + \xi_0 \]

\[ v_7 = \pm k \sqrt{\frac{\alpha(c^2-c_0^2)}{2\beta^3}} \frac{dn \xi_7}{\sqrt{m^2 - 1 + emcn \xi_7}} \exp[i(sx - (k^2(m^2 - \frac{1}{2}) + s^2)\alpha t)] \]

\[ u_7 = \frac{ak^2}{2\beta} \left( \frac{dn \xi_7}{\sqrt{m^2 - 1 + emcn \xi_7}} \right)^2 \]

\[ \xi_7 = k(x - ct) + \xi_0 \]

\[ v_8 = \left[ \mp km \sqrt{\frac{\alpha(c^2-c_0^2)(1-m^2)}{2\beta^3}} \pm k \sqrt{\frac{2 \alpha(c^2-c_0^2)(1-m^2)}{1+sn \xi_8 + cn \xi_8}} \right] \exp[i(sx - \frac{1}{2}(s^2 - k^2(1 + m^2))\alpha t)] \]

\[ u_8 = \left[ \mp km \sqrt{\frac{\alpha(1-m^2)}{2\beta}} \pm k^2 \sqrt{\frac{2 \alpha(1-m^2)}{1+sn \xi_8 + cn \xi_8}} \right]^2 \]

\[ \xi_8 = k(x - ct) + \xi_0 \]

Case 9

\[ l = 0, p = q = r = 1, \omega = (s^2 + k^2(m^2 - \frac{1}{2}))\alpha, \]

\[ c_0 = \pm k \sqrt{\frac{\alpha(c^2-c_0^2)}{2\beta^3}}, c_1 = c_2 = c_4 = 0, c_3 = \pm k \sqrt{\frac{2 \alpha(c^2-c_0^2)}{\beta^3}} \]

Case 10

\[ \omega = (k^2(1 - \frac{1}{2}m^2) + s^2)\alpha, q = r = 0, c_0 = c_1 = c_4 = 0, \]

\[ c_2 = km \sqrt{\alpha(p^2+(m^2-1)^2)(c^2-c_0^2)} / \beta^3, c_3 = \varepsilon km \sqrt{\frac{-\alpha(p^2-(\frac{1}{2}^2)(c^2-c_0^2))}{2\beta^3}} \]

We acquire the following Jacobi elliptic function solutions

\[ v_9 = \left[ \pm k \sqrt{\frac{\alpha(c^2-c_0^2)}{2\beta^3}} + \pm k \sqrt{\frac{2 \alpha(c^2-c_0^2)(1-m^2)}{1+sn \xi_9 + cn \xi_9}} \right] \exp[i(sx - (s^2 + k^2(m^2 - \frac{1}{2}))\alpha t)] \]

\[ u_9 = \left[ \pm k \sqrt{\frac{\alpha(c^2-c_0^2)}{2\beta^3}} + \pm k \sqrt{\frac{2 \alpha(c^2-c_0^2)(1-m^2)}{1+sn \xi_9 + cn \xi_9}} \right]^2 \]

\[ \xi_9 = k(x - ct) + \xi_0 \]

\[ v_{10} = km \sqrt{\frac{\alpha(c^2-c_0^2)}{2\beta^3}} \left( \frac{\sqrt{p^2+(m^2-1)^2}}{1+sn \xi_{10} + emcn \xi_{10}} + \sqrt{p^2-p^2cn \xi_{10}} \right) \exp[i(sx - (k^2(1 - \frac{1}{2}m^2) + s^2)\alpha t)] \]

\[ u_{10} = \frac{ak^2m^2}{2\beta} \left( \frac{\sqrt{p^2+(m^2-1)^2}}{1+sn \xi_{10} + emcn \xi_{10}} \right)^2 \]

\[ \xi_{10} = k(x - ct) + \xi_0 \]

**Remark 1:** Here \( u_2, v_2; u_4, v_4; u_5, v_5; u_7, v_7; u_{10}, v_{10} \) are all agree with the solutions \( u_5, v_5; u_7, v_7; u_{11}, v_{11}; u_{12}, v_{12}; u_{23}, v_{23} \) in Ref. [16], which contain the result of (13) (27); (18) (32); (19); (25) (39) in Ref. [15], for example, if we let \( p = l, r = 1 \) in \( u_{10}, v_{10} \) we can get the result of (25) (39) in Ref. [15].

**Remark 2:** It is notable that the other five types of solutions we obtained here to systems (6) are not shown in the previous literature. Some of them are degenerated to the correspoing solitary wave solutions and triangle function solutions in the limit cases when \( m \to 1 \) or \( m \to 0 \).
4 Conclusion

In this paper, we succeed to propose an approach for finding new exact solutions for nonlinear evolution equations by constructing the four new types of Jacobian elliptic functions. By using this method and computerized symbolic computation, we have found seventeen types of exact solutions for the Zakharov equations. More importantly, our method is much simple and powerful to find new solutions to various kinds of nonlinear evolution equations, such as KdV equation, mKdV equation, Boussinesq equation, etc. We believe that this method should play an important role for finding exact solutions in the mathematical physics.

References


