The Alternating Direction Iterative of Static Electromagnetic Field for Axial Symmetric Charge Distribution

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Abstract

In this paper, the problems of calculating and solution about electromagnetic fields were studied. An accurate theory model was established which can apply in high-performance numerical calculation. The new calculating method was proposed using Maxwell’s equations in the circumstances of that electromagnetic field is static, contains axial symmetric charge distribution, finite, differentiable and integrable. This calculating method was called alternating iterative method. The expression form was given on the basis of the calculating method. The electromagnetic field outside of axis can be expressed as each order derivative and one-dimensional integral of electromagnetic field which on the symmetric axis. By means of this method, the expression form is progression, which is the best form use for computer to conduct approximate calculation. It is very easy, quickly and accurately to carry out numerical calculation by using this method. The calculation method has important theoretical significance and broad prospect of application.

Keywords: Alternating Direction Iterative; Charge Distribution; Electromagnetic Field; Static; Axial Symmetry

1 Introduction

Actually, the propagation and detection, or calculation of electromagnetic field has been attended widely. In the process, scientists gave fast multipole method [1], implicit marching-on-in-time method [2], etc.

Furthermore, the phenomenon of axial symmetry is also common in the nature. Solving axial symmetric magnetic field is a problem that we often meet in the electromagnetic field theory and the practical application. If it is highly axial symmetric, that is, the magnetic field is

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only a function of cylindrical coordinates $r$, it can be solved directly by using Gauss’s theorem, loop theorem and axial symmetry. Otherwise, if it is normally axial symmetric, that is, the magnetic field is relative to both cylindrical coordinates $r$ and $z$, it is very difficult to get its generally analytic solutions. In this case, we can perform approximate calculation by using computers. Yet, owing to the complexity of problems and improper handling of them at times, the approximate solutions are not easy to approach exact solutions, or it takes a long time to get more ideal approximate solutions, or even it is quite difficult to perform approximate calculation. Along with the high performance of electronic computers in recent years, numerical simulation of electromagnetic fields have been prosperous. In the process of researches and calculations people have presented many methods, such as multilevel sparse-matrix canonical-grid method [3], time domain finite difference method [4], impedance approximate method [5], etc.

To optimize algorithm and improve computational efficiency, this paper built a precise theoretical model and gave a computational method with high performance. In the circumstances of the axial symmetric, static and passive magnetic field which is limited and differentiable on the symmetric axis, this paper gives a new kind of computing method and its results in the series form by using Maxwell’s equations and calculus. That is, the out-of-axis magnetic field can be determined by the magnetic field on the symmetric axis and its different order differential coefficient. We have to consider the electric field emitted by an electric dipole as an example and apply the method to computers for performing approximate calculation. The procedure and results of approximate calculation state that the results in the series form are the most ideal results for computer to perform approximate calculation with extremely high degree of accuracy.

The new computing method given by this paper is very important in theory and application since signal detection and calculation is an important part in scientific research and engineering. Along with the progress of science and technology, scientific researchers invented many measuring or detecting devices [6]. They can directly measure or detect signals by using them [7]. On the other hand, the researchers did in-depth theoretical studies in the analysis and presumption of the signal components [8].

In fact, many signals can be described as electromagnetic wave. So the problems become the calculation or detection of electromagnetic wave. As known, in some cases, measuring devices can not reach the regions of some signals because of limitation. So this paper gives a new kind of computing method and results to calculate or detect signals. That is, the magnetic field on one point can be calculated, detected or presumed with another point. As you know, a magnetic field can be regarded as a static state approximately if the signal changes slowly. Then, to the static and passive field, there is certain theoretical and applied value for the study how to solve the magnetic field at one point by calculating or detecting another point. To optimize algorithm and improve computational efficiency, this writer built a precise theoretical model and gave a computational method with high performance. Not long ago, the authors studied the electric field generated by a point charge, the fifth and sixth power of charge distribution in sphere symmetry using the method [9-13]. Under the circumstances of axial symmetric and without charge current distribution and static electromagnetic field that is limited and differentiable on the symmetric axis, using Maxwell’s equations and calculus gives a kind of evolutionary computing method and its results in the series form. Applying this method, out-of-axis static magnetic field can be described by the static magnetic field on the symmetric axis and its different order derivatives [14]. This paper is its further development. Under the circumstances of contains axial symmetric charge distribution and static electromagnetic field that is limited and differentiable on the symmetric axis, using Maxwell’s equations and calculus gives a kind of evolutionary computing method and
its results in the series form. Applying this method, out-of-axis static electromagnetic field can be described by the static electromagnetic field on the symmetric axis and its different order derivatives.

2 The Alternating Iteration of Static Electromagnetic Field for Axial Symmetric Charge Distribution

As is known to all, the differential form of Maxwell’s equations is

\[ \nabla \times \vec{H} = \dot{\vec{J}} + \frac{\partial \vec{D}}{\partial t} \]  
\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]  
\[ \nabla \cdot \vec{D} = \rho \]  
\[ \nabla \cdot \vec{B} = 0 \]

We choose cylindrical coordinates \( r, \varphi, z \), and the \( z \) axis is the symmetric axis of the electromagnetic field. In the circumstances of vacuum, static state, and contains axial symmetric charge distribution, applying Maxwell’s equations and axial symmetry, the equations of the electromagnetic field to follow are

\[ -\frac{\partial B_\varphi}{\partial z} \hat{e}_r + \left( \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} \right) \hat{e}_\varphi + \frac{1}{r} \frac{\partial (r B_\varphi)}{\partial r} \hat{e}_z = 0 \]  
\[ -\frac{\partial E_\varphi}{\partial z} \hat{e}_r + \left( \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} \right) \hat{e}_\varphi + \frac{1}{r} \frac{\partial (r E_\varphi)}{\partial r} \hat{e}_z = 0 \]  
\[ \frac{1}{r} \frac{\partial (r E_r)}{\partial r} + \frac{\partial E_z}{\partial z} = \frac{1}{\varepsilon_0} \rho \]  
\[ \frac{1}{r} \frac{\partial (r B_r)}{\partial r} + \frac{\partial B_z}{\partial z} = 0 \]

From the (5), (6), (7), and (8), we have

\[ E_r(r, z) = \frac{1}{\varepsilon_0} \frac{1}{r} \int_0^r \int \rho(r, z) r dr - \frac{1}{r} \int_0^r \frac{\partial E_z(r, z)}{\partial z} r dr \]  
\[ E_z(r, z) = E(z) + \int_0^r \frac{\partial E_r(r, z)}{\partial z} dr \]  
\[ E_\varphi(r, z) = 0 \]  
\[ B_r(r, z) = -\frac{1}{r} \int_0^r \frac{\partial B_z(r, z)}{\partial z} r dr \]  
\[ B_z(r, z) = B(z) + \int_0^r \frac{\partial B_r(r, z)}{\partial z} dr \]
\[ B_\varphi(r, z) = 0 \quad (14) \]

\[ E(z) = E_z(0, z) \text{ and } B(z) = B_z(0, z) \text{ in the (10) and (13). The equations (9), (10), (11), (12), (13) and (14) show that the axial symmetric, contains axial symmetric charge distribution and static state electromagnetic field } E_\varphi(r, z) \text{ and } B_\varphi(r, z) \text{ are identically vanishing. However, because of the mutual nestification of } E_r(r, z), E_z(r, z), B_r(r, z) \text{ and } B_z(r, z), \text{ none of them can be solved directly. In order to solve } E_r(r, z), E_z(r, z), B_r(r, z) \text{ and } B_z(r, z), \text{ we suggest the new computing method ( alternating iteration method ). That is, one should choose a proper initial value and make use of the (9), (10), (12), and (13) to perform alternating iteration. Obviously, approximate solutions are given by the alternating iteration of finite degrees and times. With increasing the degrees and times of alternating iteration, the solutions are more exact. After the alternating iteration of infinite degrees and times, the exact solutions of the out-of-axis electromagnetic field can be given. Here, we choose } E_z(r, z)'s \text{ and } B_z(r, z)'s \text{ value on the symmetric axis as the initial value, that is}

\[ E_0z(r, z) = E_z(0, z) = E(z) \quad (15) \]

\[ B_0z(r, z) = B_z(0, z) = B(z) \quad (16) \]

\[ E(z) \text{ and } B(z) \text{ were also expressed into}
\]

\[ E(z) = \frac{(-1)^0}{(0!)^22^{0}} E_z^{(2\times0)}(z)(r)^{2\times0} \quad (17) \]

\[ B(z) = \frac{(-1)^0}{(0!)^22^{0}} B_z^{(2\times0)}(z)(r)^{2\times0} \quad (18) \]

Substituting the (15) into the (9) and substituting the (16) into the (12), we can get

\[ E_0r(r, z)g = \frac{1}{\varepsilon_0} (0,1) \rho(r, z) + \frac{(-1)^{0+1}}{(0+1)!0!2^{0+0+1}} E_z^{(2\times0+1)}(z)(r)^{2\times0+1} \]  

\[ B_0r(r, z) = \frac{(-1)^{0+1}}{(0+1)!0!2^{0+0+1}} B_z^{(2\times0+1)}(z)(r)^{2\times0+1} \quad (20) \]

\[ E^{(n)}(z) \text{ is the } n \text{th derivative of } E(z) \text{ with respect to } z \text{ in the (19), and } B^{(n)}(z) \text{ is the } n \text{th derivative of } B(z) \text{ with respect to } z \text{ in the (20), i.e.}
\]

\[ E^{(n)}(z) = \frac{\partial}{\partial z} E^{(n-1)}(z) \quad (21) \]

\[ B^{(n)}(z) = \frac{\partial}{\partial z} B^{(n-1)}(z) \quad (22) \]

And that

\[ (0,1) \rho(r, z) = \frac{1}{r} \int_0^r \rho(r, z) r dr \quad (23) \]

Substituting the (19) into the (10), and substituting the (20) into the (13), we get

\[ E_{1z}(r, z) = \frac{(-1)^0}{(0!)^22^{0}} E_z^{(2\times0)}(z)(r)^{2\times0} + \frac{(-1)^1}{(1!)^22^{1}} E_z^{(2\times1)}(z)(r)^{2\times1} + (-1)^{1+1} \frac{1}{\varepsilon_0} (1,1) \rho^{(2\times1-1)}(r, z) \]  

\[ \text{(24)} \]
Substituting the (30) into the (10), and substituting the (31) into the (13), we get

\[
B_{1z}(r, z) = \frac{(-1)^0}{(0!)^2 2^x 0} B^{(2x0)}(z)(r)^{2x0} + \frac{(-1)^1}{(1!)^2 2^x 1} B^{(2x1)}(z)(r)^{2x1}
\]  

(25)

In the formula (24), \((1, 1)\rho^{(2x1-1)}(r, z)\) is expressed as the following formula

\[
(1, 1)\rho^{(2x1-1)}(r, z) = \int_0^r (0, 1)\rho^{(2x1-1)}(r, z)dr = \int_0^r \frac{\partial}{\partial z} [(0, 1)\rho(r, z)]dr
\]  

(26)

In the above formula, \((m, n)\rho^{(l)}(r, z)\) is the \(l\)th derivative of \(\rho(r, z)\) with respect to \(z\), and the \((m + n)\)th integral of \(\rho(r, z)\) with respect to \(r\), i.e.

\[
(m, n)\rho^{(l)}(r, z) = \frac{\partial}{\partial z} (m, n)\rho^{(l-1)}(r, z)
\]  

(27)

\[
(m, n)\rho^{(l)}(r, z) = \int_0^r (m-1, n)\rho^{(l)}(r, z)dr
\]  

(28)

\[
(m, n)\rho^{(l)}(r, z) = \frac{1}{r} \int_0^r (m, n-1)\rho^{(l)}(r, z)rdr
\]  

(29)

Substituting the (24) into the (9), and substituting the (25) into the (12), we obtain

\[
E_{1r}(r, z) = \frac{(-1)^0}{(0!)^2 2^x 0} E^{(2x0)}(z)(r)^{2x0} + \frac{(-1)^1}{(1!)^2 2^x 1} E^{(2x1)}(z)(r)^{2x1} + \frac{1}{\varepsilon_0} (0, 1)\rho(r, z) + (-1)^1 \frac{1}{\varepsilon_0} (1, 1)\rho^{(1)}(r, z)
\]  

(30)

\[
B_{1r}(r, z) = \frac{(-1)^0}{(0!)^2 2^x 0} B^{(2x0)}(z)(r)^{2x0} + \frac{(-1)^1}{(1!)^2 2^x 1} B^{(2x1)}(z)(r)^{2x1} + \frac{1}{\varepsilon_0} (0, 1)\rho(r, z) + (-1)^1 \frac{1}{\varepsilon_0} (1, 1)\rho^{(1)}(r, z)
\]  

(31)

Substituting the (30) into the (10), and substituting the (31) into the (13), we get

\[
E_{2z}(r, z) = \frac{(-1)^0}{(0!)^2 2^x 0} E^{(2x0)}(z)(r)^{2x0} + \frac{(-1)^1}{(1!)^2 2^x 1} E^{(2x1)}(z)(r)^{2x1} + \frac{(-1)^2}{(2!)^2 2^x 2} E^{(2x2)}(z)(r)^{2x2} + (-1)^{1+1} \frac{1}{\varepsilon_0} (1, 1)\rho^{(2x1-1)}(r, z) + (-1)^{2+1} \frac{1}{\varepsilon_0} (2, 2)\rho^{(2x2-1)}(r, z)
\]  

(32)

\[
B_{2z}(r, z) = \frac{(-1)^0}{(0!)^2 2^x 0} B^{(2x0)}(z)(r)^{2x0} + \frac{(-1)^1}{(1!)^2 2^x 1} B^{(2x1)}(z)(r)^{2x1} + \frac{(-1)^2}{(2!)^2 2^x 2} B^{(2x2)}(z)(r)^{2x2}
\]  

(33)

Substituting the (32) into the (9), and substituting the (33) into the (12), we obtain

\[
E_{2r}(r, z) = \frac{(-1)^0}{(0!)^2 2^x 0} E^{(2x0)}(z)(r)^{2x0} + \frac{(-1)^1}{(1!)^2 2^x 1} E^{(2x1)}(z)(r)^{2x1} + \frac{(-1)^2}{(2!)^2 2^x 2} E^{(2x2)}(z)(r)^{2x2} + \frac{(-1)^0}{(0!)^2 2^x 0} E^{(2x0)}(z)(r)^{2x0} + \frac{(-1)^1}{(1!)^2 2^x 1} E^{(2x1)}(z)(r)^{2x1} + \frac{(-1)^2}{(2!)^2 2^x 2} E^{(2x2)}(z)(r)^{2x2}
\]  

\[
+ (-1)^{1+1} \frac{1}{\varepsilon_0} (1, 1)\rho^{(2x1-1)}(r, z) + (-1)^{2+1} \frac{1}{\varepsilon_0} (2, 2)\rho^{(2x2-1)}(r, z)
\]  

(34)

\[
B_{2r}(r, z) = \frac{(-1)^0}{(0!)^2 2^x 0} B^{(2x0)}(z)(r)^{2x0} + \frac{(-1)^1}{(1!)^2 2^x 1} B^{(2x1)}(z)(r)^{2x1} + \frac{(-1)^2}{(2!)^2 2^x 2} B^{(2x2)}(z)(r)^{2x2}
\]  

(35)
After substituting the (36) into the (10), and substituting the (37) into the (12), we obtain

\[ \begin{align*}
E_{3z}(r, z) &= \frac{(-1)^0}{(0!)^22^0} E^{(2\times0)}(z)(r)^{2\times0} + \frac{(-1)^1}{(1!)^22^1} E^{(2\times1)}(z)(r)^{2\times1} + \frac{(-1)^2}{(2!)^22^2} E^{(2\times2)}(z)(r)^{2\times2} \\
&+ \frac{(-1)^3}{(3!)^22^3} E^{(2\times3)}(z)(r)^{2\times3} + (-1)^{1+1} \frac{1}{\xi_0} (1, 1) \rho^{(2\times1-1)}(r, z) \\
&+ (-1)^{2+1} \frac{1}{\xi_0} (2, 2) \rho^{(2\times2-1)}(r, z) + (-1)^{3+1} \frac{1}{\xi_0} (3, 3) \rho^{(2\times3-1)}(r, z)
\end{align*} \]

\[ (36) \]

\[ \begin{align*}
B_{3z}(r, z) &= \frac{(-1)^0}{(0!)^22^0} B^{(2\times0)}(z)(r)^{2\times0} + \frac{(-1)^1}{(1!)^22^1} B^{(2\times1)}(z)(r)^{2\times1} + \frac{(-1)^2}{(2!)^22^2} B^{(2\times2)}(z)(r)^{2\times2} \\
&+ \frac{(-1)^3}{(3!)^22^3} B^{(2\times3)}(z)(r)^{2\times3}
\end{align*} \]

\[ (37) \]

Substituting the (36) into the (9), and substituting the (37) into the (12), we obtain

\[ \begin{align*}
E_{3r}(r, z) &= \frac{(-1)^{0+1}}{(0+1)!2^{0+1}} E^{(2\times0+1)}(z)(r)^{2\times0+1} + \frac{(-1)^{1+1}}{(1+1)!2^{1+1}} E^{(2\times1+1)}(z)(r)^{2\times1+1} \\
&+ \frac{(-1)^{2+1}}{(2+1)!2^{2+1}} E^{(2\times2+1)}(z)(r)^{2\times2+1} + \frac{(-1)^{3+1}}{(3+1)!2^{3+1}} E^{(2\times3+1)}(z)(r)^{2\times3+1} \\
&+ \frac{1}{\xi_0} (0, 1) \rho(r, z) + (-1)^{1} \frac{1}{\xi_0} (1, 1+1) \rho^{(2\times1)}(r, z) + (-1)^{2} \frac{1}{\xi_0} (2, 2+1) \rho^{(2\times2)}(r, z) + (-1)^{3} \times \frac{1}{\xi_0} (3, 3+1) \rho^{(2\times3)}(r, z)
\end{align*} \]

\[ (38) \]

\[ \begin{align*}
B_{3r}(r, z) &= \frac{(-1)^{0+1}}{(0+1)!2^{0+1}} B^{(2\times0+1)}(z)(r)^{2\times0+1} + \frac{(-1)^{1+1}}{(1+1)!2^{1+1}} B^{(2\times1+1)}(z)(r)^{2\times1+1} \\
&+ \frac{(-1)^{2+1}}{(2+1)!2^{2+1}} B^{(2\times2+1)}(z)(r)^{2\times2+1} + \frac{(-1)^{3+1}}{(3+1)!2^{3+1}} B^{(2\times3+1)}(z)(r)^{2\times3+1}
\end{align*} \]

\[ (39) \]

After \( N \)th iteration, we can get

\[ \begin{align*}
E_{Nz}(r, z) &= \sum_{n=0}^{N} \frac{(-1)^n}{(n!)^22^{2n}} E^{(2n)}(z)(r)^{2n} + \frac{1}{\xi_0} \sum_{n=1}^{N} (-1)^{n+1(n,n)} \rho^{(2n-1)}(r, z) \\
&+ \frac{1}{\xi_0} \sum_{n=1}^{N} (-1)^{n(n,n+1)} \rho^{(2n)}(r, z)
\end{align*} \]

\[ (40) \]

\[ \begin{align*}
B_{Nz}(r, z) &= \sum_{n=0}^{N} \frac{(-1)^n}{(n!)^22^{2n}} B^{(2n)}(z)(r)^{2n}
\end{align*} \]

\[ (41) \]

\[ \begin{align*}
E_{Nr}(r, z) &= \sum_{n=0}^{N} \frac{(-1)^{n+1}}{(n+1)!2^{2n+1}} E^{(2n+1)}(z)(r)^{2n+1} + \frac{1}{\xi_0} (0, 1) \rho(r, z) \\
&+ \frac{1}{\xi_0} \sum_{n=1}^{N} (-1)^{n(n,n+1)} \rho^{(2n)}(r, z)
\end{align*} \]

\[ (42) \]

\[ \begin{align*}
B_{Nr}(r, z) &= \sum_{n=0}^{N} \frac{(-1)^{n+1}}{(n+1)!2^{2n+1}} B^{(2n+1)}(z)(r)^{2n+1}
\end{align*} \]

\[ (43) \]
The rest may be deduced by analogy. The exact solutions of $E_z(r, z)$, $B_z(r, z)$, $E_r(r, z)$ and $B_r(r, z)$ can be given by the alternating iteration of infinite degree. That is

$$E_z(r, z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2 2^{2n}} E^{(2n)}(z)(r)^{2n} + \frac{1}{\varepsilon_0} \sum_{n=1}^{\infty} (-1)^{n+1} \rho^{(2n-1)}(r, z)$$

$$B_z(r, z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2 2^{2n}} B^{(2n)}(z)(r)^{2n}$$

$$E_r(r, z) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)! n! 2^{2n+1}} E^{(2n+1)}(z)(r)^{2n+1} + \frac{1}{\varepsilon_0} \sum_{n=1}^{\infty} (-1)^{n(n+1)} \rho^{(2n)}(r, z)$$

$$B_r(r, z) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)! n! 2^{2n+1}} B^{(2n+1)}(z)(r)^{2n+1}$$

3 Conclusion

According to the study, we can get a conclusion. In the circumstances of the static, axial symmetric, contains charge distribution, finite, differentiable, and integrable, this paper provides a new kind of computing method (alternating iteration method) and its results in the series form by making use of Maxwell equations and calculus. That is, the out-of-axis electromagnetic field can be determined by the electromagnetic field on the symmetric axis and its different order derivatives. It has been exact enough, and it is very easy to perform approximate calculation with quite high speed. Especially to those problems of some out-of-axis areas that are not easy to measure or get exact solutions from theory, it is easy, quick and exact to solve them by using the method and results given by this paper.

References


